Direct approach to the study of soliton perturbations

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A direct approach in studying the soliton perturbations of nonlinear evolution equations has been developed. It is based on the method of the derivative expansions (for linearization of the perturbed equations), and the separation of variables (for solution of the linearized equations). It differs substantially from the past direct methods. The apparent advantage of this approach is that it relies on no knowledge of the inverse scattering transform. Besides, it is very concise and easy to understand. As an example, we use it to study the perturbed Korteweg–de Vries equation. The results we obtained agree with what other authors have found. [S1063-651X(96)12311-4]

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I. INTRODUCTION

The physical situations that give rise to the standard soliton equations tend to be highly idealized. In more realistic situations, we derive equations that differ slightly from the standard ones by small additional terms that are called perturbations. One of the most powerful techniques in dealing with these cases is based on the inverse scattering transform (IST) [1–5], which requires the unperturbed equations to be exactly sovable by the IST. Apart from the sophistication and the restriction of the range of applications, this technique is inconvenient for one who is not familiar with IST.

Another alternative way to study soliton perturbations is the direct method. Since it was first developed by Ostrovskii and his colleagues [6,7], many other authors have used various direct methods in their works [8–12]. Here we would roughly like to describe the characteristic features of this method. Original perturbed nonlinear equations are usually linearized by expanding their solutions about the unperturbed solutions for the purpose of taking the perturbations into account directly [11]. At the same time, derivative expansion for the time is always needed to eliminate the potential secular terms. The basic technical ingredient of this method is to find eigenfunctions of a linear operator associated with the linearized equation. The perturbations and the first-order corrections are expanded into a series based on a complete set of those eigenfunctions. Then the time dependence of the soliton parameters and the first-order corrections are readily available.

Implementation of this idea can be seen, for example, in Refs. [16,17] in which Kaup used a direct method to study sine-Gordon (SG) equations by employing some approximations. Several other authors have also used direct methods in their study of soliton perturbations by adopting a quasistationary assumption [18–20]. These works have no need to rely on the techniques of IST. Important contributions to the direct perturbation method for the nonlinear Schrödinger (NLS) equation have been made by Keener and McLaughlin [15]. Their approach is based on a Green's function and ''two timing'' procedure. The modulations in the speeds and in the locations of the soliton can be computed directly from the soliton wave form. Representation of the Green's function necessary for the calculation of the first-order correction is found in the framework of the IST. Herman's excellent work should be noted here [13,14]. In his papers, the method of multiple time scales is used to linearize the perturbed nonlinear evolution equations. After that, the linearized equations are solved by inverting the corresponding operators. For doing that, the associated ''Lax pair'' of the given evolution equation is used to establish relations between the eigenfunctions of the linear operator and that of the associated eigenvalue problem of the given evolution equation. Then a complete set of eigenfunctions needed for the inversion of the operator is obtained.

We find a direct method to study soliton perturbations. This method relies on no tools of IST. Besides, no approximations are employed. It is very concise and easy to understand. In this paper, we apply it to the perturbed Korteweg-de Vries (KdV) equation in an attempt to demonstrate it. Now we describe the procedure roughly. As usual the derivative expansion method is used to linearize the perturbed soliton equations. By introducing a coordinate system moving with the soliton, namely, by choosing the soliton phase z as a new independent variable, z and the time variable t are separated in different terms of the linearized equations. Obviously, these equations can be solved in principle by the method of separation of variables. The key to the problem is how to find the eigenfunctions of an appropriate operator $\hat{L}$ and its adjoint $\hat{L}^{\dagger}$, based on which, we can construct two complete sets of states as the bases of a perturbation expansion. Fortunately, the eigenfunctions can be derived in a general way, and the completeness and orthogonality of the bases can be proved through a straightforward calculation with the aid of the residue theorem. It must be pointed out that since our eigenfunctions are time independent, our approach is simpler than the other direct method, and it is particularly convenient for studying the time-dependent perturbations. In this paper, the damping KdV equation and KdV-Burgers equation have been discussed in detail as two important examples. The results are consistent with those obtained by other authors. We believe

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that this approach can be extended to deal with NLS equations, SG equations and perhaps some other nonlinear evolution equations.

II. LINEARIZATION OF KdV EQUATION

Let us consider the perturbed KdV equation

$$u_t+6uu_x+u_{xxx}=\epsilon R[u], \tag{1}$$

where the subscripts stand for partial differentiation with respect to time t and space x, ϵ is a small positive constant measuring the weakness of the perturbation ($0<\epsilon\ll 1$), and the perturbation term $R[u]$ is a known function of u, u_x, u_{xx} . . . . It is well known that the standard KdV equation has a single-soliton solution

$$u=2a^2\ \mathrm{sech}^2 a(x-4a^2t-x_0), \tag{2}$$

where a and x_0 are two real parameters that determine the soliton height (width and velocity as well) and initial position, respectively. Since what we study in this paper is the effects of perturbation on a single soliton, Eq. (1) is subject to the initial condition

$$u(x,o)=2a^2\ \mathrm{sech}^2 a(x-x_0). \tag{3}$$

At first, we linearize Eq. (1) following the lines of Refs. [11, 12]. The independent variable t is transformed into several variables by

$$t_n=\epsilon^n t, \quad n=0,1,2,\ldots, \tag{4}$$

where each t_n is an order of ϵ smaller than the previous time. Thus the time derivatives should be replaced by the expansion

$$\partial_t=\partial_{t_0}+\epsilon\partial_{t_1}+\epsilon^2\partial_{t_2}+\cdots\ . \tag{5}$$

At the same time, u and $R[u]$ are expanded in an asymptotic series

$$u=u^{(0)}+\epsilon u^{(1)}+\epsilon^2 u^{(2)}+\cdots\ , \tag{6}$$

$$R[u]=R^{(1)}[u^0]+\epsilon R^{(2)}[u^{(0)},u^{(1)}]+\cdots\ . \tag{7}$$

Substituting Eqs. (5)–(7) into Eq. (1), and equating the coefficients of each power of ϵ, we obtain the following approximation equations of different orders:

$$u_{t_0}^{(0)}+6u^{(0)}u_x^{(0)}+u_{xxx}^{(0)}=0, \tag{8}$$

$$u_{t_0}^{(1)}+6u^{(0)}u_x^{(1)}+6u_x^{(0)}u^{(1)}+u_{xxx}^{(1)}=R^{(1)}[u^0]-u_{t_1}^{(0)}, \tag{9}$$

$$\begin{aligned}&u_{t_0}^{(2)}+6u^{(0)}u_x^{(2)}+6u_x^{(0)}u^{(2)}+u_{xxx}^{(2)}\\&\quad=R^{(2)}[u^{(0)},u^{(1)}]-u_{t_2}^{(0)}-u_{t_1}^{(1)}-6u^{(1)}u_x^{(1)}\ \ldots\ .\end{aligned} \tag{10}$$

Meanwhile, the initial condition (3) should be replaced by

$$\begin{aligned}&u^{(0)}(x,0)=2a^2\ \mathrm{sech}^2 a(x-x_0),\\&u^{(n)}(x,0)=0 \quad \text{for } n=1,2,\ldots .\end{aligned} \tag{11}$$

The zeroth-order approximation equation (8) is just the standard KdV equation. It has a single-soliton solution that is formally the same as Eq. (2):

$$u^{(0)}(x,t_0)=2a^2\ \mathrm{sech}^2 z, \quad z=a(x-\xi), \quad \xi_{t_0}=4a^2. \tag{12}$$

Due to perturbation, the soliton parameters a and ξ are now supposed to be functions of the slow time variables $t_1, t_2,\ldots$, but a is independent of t_0, and the dependence of ξ on t_0 is given by $\xi_{t_0}=4a^2$. It follows from Eq. (12) that

$$u_{t_n}^{(0)}=4aa_{t_n}\phi_1(z)+4a^3\xi_{t_n}\phi_2(z), \quad n=1,2,\ldots, \tag{13}$$

where

$$\phi_1(z)=\frac{1}{4a}\frac{\partial}{\partial a}u^{(0)}=(1-z\ \tanh z)\mathrm{sech}^2 z, \tag{14}$$

$$\phi_2(z)=\frac{1}{4a^3}\frac{\partial}{\partial \xi}u^{(0)}=\tanh z\ \mathrm{sech}^2 z. \tag{15}$$

Then the linearized KdV equations (9) and (10), together with the appropriate initial conditions (11) are reduced into the following form with the aid of Eq. (13):

$$\begin{aligned}u_{t_0}^{(1)}+a^3\hat{L}u^{(1)}&=F^{(1)}(z)\\&\equiv R^{(1)}-4aa_{t_1}\phi_1(z)-4a^3\xi_{t_1}\phi_2(z),\\u^{(1)}(z,0)&=0,\end{aligned} \tag{16}$$

$$\begin{aligned}u_{t_0}^{(2)}+a^3\hat{L}u^{(2)}&=F^{(2)}(z)\\&\equiv R^{(2)}-4aa_{t_2}\phi_1(z)-4a^3\xi_{t_2}\phi_2(z)-u_{t_1}^{(1)}\\&\quad-6au^{(1)}u_z^{(1)},\\u^{(2)}(z,0)&=0,\end{aligned} \tag{17}$$

where z is the space coordinate in a system moving with the soliton, and $\hat{L}$ is a linear differential operator defined as follows:

$$\hat{L}=\frac{d^3}{dz^3}+(12\ \mathrm{sech}^2 z-4)\frac{d}{dz}-24\ \tanh z\ \mathrm{sech}^2 z. \tag{18}$$

It is apparent that $\hat{L}$ is not self-adjoint, and its adjoint is

$$\hat{L}^{\dagger}=\frac{d^3}{dz^3}+(12\ \mathrm{sech}^2 z-4)\frac{d}{dz}. \tag{19}$$

III. EIGENVALUE PROBLEM

Now we derive the solutions of the linearized KdV equations (16) and (17) by the separation of variables. It is known that the key to this problem is to solve the eigenvalue problem of operator $\hat{L}$ and its adjoint $\hat{L}^{\dagger}$,

$$\hat{L}\phi=\lambda\phi. \tag{20}$$

$$\hat{L}^{\dagger}\psi=\lambda'\psi. \tag{21}$$

In Appendix A, the eigenfunction of Eq. (20) for continuous eigenvalue $\lambda=-ik(k^2+4)$, $-\infty<k<\infty$ is derived in a general way,

$$\phi(z,k)=\frac{1}{\sqrt{2\pi}k(k^2+4)}[k(k^2+4)+4i(k^2+2)\tanh z -8k\tanh^2 z-8i\tanh^3 z]e^{ikz}, \tag{22}$$

and it is easy to check directly that $\hat{L}$ also has a eigenfunction for discrete eigenvalue $\lambda=0$,

$$\phi_2(z)=\tanh z\ \mathrm{sech}^2 z. \tag{23}$$

Similarly, for the associated eigenvalue problem (21), we have

$$\psi(z,k)=\frac{1}{\sqrt{2\pi}(k^2+4)}[k^2-4ik\tanh z-4\tanh^2 z]e^{-ikz} \tag{24}$$

for continuous eigenvalue $\lambda'=ik(k^2+4)$, $-\infty<k<\infty$, and

$$\psi_1(z)=\mathrm{sech}^2 z \tag{25}$$

for the discrete eigenvalue $\lambda'=0$.

There is a function $\phi_1(z)=(1-z\tanh z)\mathrm{sech}^2 z$ that satisfies the equation $L\phi_1(z)=-8\phi_2(z)$. It turns out to be a standard problem and is well known that $\phi_1(z)$ needs to be included in the completeness relationship. In the meantime, for its adjoint states, another function $\psi_2(z)=\tanh z+z\,\mathrm{sech}^2 z$ needs to be included as well. The completeness of $\{\phi\}=\{\phi(z,k),\phi_j(z);j=1,2\}$ and $\{\psi\}=\{\psi(z,k),\psi_j(z);j=1,2\}$ is expressed as

$$\mathrm{P}\int_{-\infty}^{\infty}\phi(z,k)\psi(z',k)dk+\sum_{j=1}^{2}\phi_j(z)\psi_j(z')=\delta(z-z'), \tag{26}$$

where P denotes the principal value of an integral, since $k=0$ is a simple pole of the integrand. The following orthogonal relations are also proved in Appendix C:

$$\int_{-\infty}^{\infty}\phi(z,k)\psi(z,k')dz=\delta(k-k'), \tag{27}$$

$$\int_{-\infty}^{\infty}\phi(z,k)\psi_j(z)dz=\int_{-\infty}^{\infty}\psi(z,k)\phi_j(z)dz=0,\quad j=1,2, \tag{28}$$

$$\int_{-\infty}^{\infty}\phi_j(z)\phi_l(z)dz=\delta_{jl},\quad j,l=1,2. \tag{29}$$

Based on this complete basis, any function $F(z)$ can be expanded in a generalized Fourier integral as follows:

$$F(z)=\mathrm{P}\int_{-\infty}^{\infty}f(k)\phi(z,k)dk+\sum_{j=1}^{2}f_j\phi_j(z), \tag{30}$$

where the generalized Fourier coefficients can be obtained by the orthogonality

$$f(k)=\int_{-\infty}^{\infty}F(z)\psi(z,k)dz, \tag{31}$$

$$f_j=\int_{-\infty}^{\infty}F(z)\psi_j(z)dz,\quad j=1,2. \tag{32}$$

IV. EFFECTS OF PERTURBATION ON A SOLITON

Now we return to Eqs. (16) and (17). First, let us consider the first-order approximation equation (16). To solve this initial-value problem by separation of variables, we expand $u^{(1)}$ and $F^{(1)}$ on the basis $\{\phi\}$ as

$$u^{(1)}(z,t_0)=\mathrm{P}\int_{-\infty}^{\infty}T^{(1)}(t_0,k)\phi(z,k)dk+\sum_{j=1}^{2}T_j^{(1)}(t_0)\phi_j(z), \tag{33}$$

$$F^{(1)}\equiv R^{(1)}-4aa_{t_1}\phi_1(z)-4a^3\xi_{t_1}\phi_2(z) =\mathrm{P}\int_{-\infty}^{\infty}f^{(1)}(k)\phi(z,k)dk+\sum_{j=1}^{2}f_j^{(1)}\phi_j(z). \tag{34}$$

The expansion coefficients of $F^{(1)}(z)$ are obtained by employing the orthogonality relations Eqs. (27)–(29) as follows:

$$f^{(1)}(k)=\int_{-\infty}^{\infty}R^{(1)}[u^{(0)}(z)]\psi(z,k)dz, \tag{35}$$

$$f_1^{(1)}=-4aa_{t_1}+\int_{-\infty}^{\infty}R^{(1)}[u^{(0)}(z)]\psi_1(z)dz, \tag{36}$$

$$f_2^{(1)}=-4a^2\xi_{t_1}+\int_{-\infty}^{\infty}R^{(1)}[u^{(0)}(z)]\psi_2(z)dz. \tag{37}$$

We now substitute Eqs. (33) and (34) into Eq. (16), and note that $\hat{L}\phi(z,k)=-ik(k^2+4)\phi(z,k)$, $\hat{L}\phi_2(z)=0$, and $\hat{L}\phi_1(z)=-8\phi_2(z)$. Then Eq. (16) may be rewritten as

$$\mathrm{P}\int_{-\infty}^{\infty}[\dot{T}^{(1)}(t_0,k)-ik(k^2+4)a^3T^{(1)}(t_0,k)]\phi(z,k)dk +\dot{T}_1^{(1)}(t_0)\phi_1(z)+\dot{T}_2^{(1)}(t_0)\phi_2(z)-8a^3T_1^{(1)}(t_0)\phi_2(z) =\mathrm{P}\int_{-\infty}^{\infty}f^{(1)}(k)\phi(z,k)dk+\sum_{j=1}^{2}f_j^{(1)}\phi_j(z), \tag{38}$$

where the overdot signifies derivative with respect to t_0. Multiplying Eq. (38) by $\psi(z,k)$, $\psi_1(z)$, and $\psi_2(z)$ successively, and then integrating over z, we obtain the following ordinary differential equations:

$$\dot{T}^{(1)}(t_0,k)-ik(k^2+4)a^3T^{(1)}(t_0,k)=f^{(1)}(k),\quad T^{(1)}(0,k)=0, \tag{39}$$

$$\dot{T}_1^{(1)}(t_0)=f_1^{(1)},\quad T_1^{(1)}(0)=0, \tag{40}$$

$$\dot{T}_2^{(1)}(t_0)-8a^3T_1^{(1)}(t_0)=f_2^{(1)},\quad T_2^{(1)}(0)=0, \tag{41}$$

where the zero-initial conditions $T^{(1)}(0,k) = T_1^{(1)}(0) = T_2^{(1)}(0) = 0$ come from the second equation in Eq. (16). Since both $f_1^{(1)}$ and $f_2^{(1)}$ are constants, Eqs. (40) and (41) will lead to the secularity. In fact, while integrating Eq. (40), it yields $T_1^{(1)} = f_1^{(1)} t_0$, which grows infinitely in time. So we must demand that

$$f_1^{(1)} = 0 \rightarrow T_1^{(1)}(t_0) \equiv 0. \tag{42}$$

Then Eq. (41) is reduced to $\dot{T}_2^{(1)}(t_0) = f_2^{(1)}$, and similarly,

$$f_2^{(1)} = 0 \rightarrow T_2^{(1)}(t_0) \equiv 0. \tag{43}$$

Now we are going to find out the effects of perturbation on the soliton. At first, inserting Eqs. (42) and (43) into Eqs. (36) and (37), we get the following two important formulas immediately:

$$a_{t_1} = \frac{1}{4a} \int_{-\infty}^{\infty} R^{(1)}[u^{(0)}] \psi_1(z) dz = \frac{1}{4a} \int_{-\infty}^{\infty} R^{(1)}[u^{(0)}] \text{sech}^2 z \, dz, \tag{44}$$

$$\xi_{t_1} = \frac{1}{4a^3} \int_{-\infty}^{\infty} R^{(1)}[u^{(0)}] \psi_2(z) dz = \frac{1}{4a^3} \int_{-\infty}^{\infty} R^{(1)}[u^{(0)}](\tanh z + z \, \text{sech}^2 z) dz. \tag{45}$$

Obviously, Eqs. (44) and (45) are the same with those obtained in other ways [11,12]. They determine how the soliton shape and position are affected by the perturbation. As to Eq. (39), it is a first-order ordinary differential equation with constant coefficients and the right-hand side is independent of t_0, so it can be solved easily in a standard method:

$$T^{(1)}(t_0,k) = -f^{(1)}(k)\{1 - \exp[ik(k^2+4)a^3 t_0]\}/ik(k^2+4)a^3. \tag{46}$$

Substituting Eq. (46) into Eq. (33), and noting that $T_1^{(1)}(t_0) = T_2^{(1)}(t_0) = 0$, we get

$$u^{(1)}(z,t_0) = -\text{P} \int_{-\infty}^{\infty} \frac{f^{(1)}(k)}{ik(k^2+4)a^3} \times \{1 - \exp[ik(k^2+4)a^3 t_0]\} \phi(z,k) dk, \tag{47}$$

where $f^{(1)}(k)$ is determined by the perturbation through Eq. (35). One can see clearly from Eq. (47) that the first-order correction $u^{(1)}(z,t_0)$ is associated with the continuous spectrum only. While combining Eq. (35) with Eq. (47), this correction can be expressed in terms of a Green's function,

$$u^{(1)}(z,t_0) = \int_{-\infty}^{\infty} G^{(1)}(z,z';t_0) R^{(1)}[u^{(0)}(z')] dz', \tag{48}$$

where the first-order Green's function $G^{(1)}(z,z';t_0)$ is defined by

$$G^{(1)}(z,z';t) = -\text{P} \int_{-\infty}^{\infty} \frac{1 - \exp[ik(k^2+4)a^3 t]}{ik(k^2+4)a^3} \times \phi(z,k) \psi(z',k) dk. \tag{49}$$

In a similar way, we can derive the higher-order effects of perturbation to the soliton. For example, the second-order effects are given as follows:

$$a_{t_1} = \frac{1}{4a} \int_{-\infty}^{\infty} \{R^{(2)}[u^{(0)},u^{(1)}] - u_{t_1}^{(1)} - 6au^{(1)}u_z^{(1)}\} \text{sech}^2 z \, dz, \tag{50}$$

$$\xi_{t_2} = \frac{1}{4a^3} \int_{-\infty}^{\infty} \{R^{(2)}[u^{(0)},u^{(1)}] - u_{t_1}^{(1)} - 6au^{(1)}u_z^{(1)}\} \times (\tanh z + z \, \text{sech}^2 z) dz, \tag{51}$$

$$u^{(2)} = -\text{P} \int_{-\infty}^{\infty} \frac{f^{(2)}(k)}{ik(k^2+4)a^3} \times \{1 - \exp[ik(k^2+4)a^3 t_0]\} \phi(z,k) dk, \tag{52}$$

where

$$f^{(2)}(k) = \int_{-\infty}^{\infty} [R^{(2)} - u_{t_1}^{(1)} - 6au^{(1)}u_z^{(1)}] \psi(z,k) dz. \tag{53}$$

A more detailed discussion is omitted here for simplicity.

V. TIME-DEPENDENT PERTURBATIONS

In this section, we show that our approach is particularly convenient for the study of the time-dependent perturbations. For this purpose, we assume that the perturbation $R(x,t)$ is a given function of x and t. In the coordinate system moving with the soliton, it is denoted by $R(z,t)$. Obviously, Eqs. (33)–(41) in Sec. IV still apply, except that coefficients $f^{(1)}(k)$ and $f_j^{(1)}$, $j=1,2$ are now time dependent, and are denoted by $f^{(1)}(t_0,k)$ and $f_j^{(1)}(t_0)$, respectively. For later convenience, we rewrite Eqs. (39)–(41) as follows:

$$\dot{T}^{(1)}(t_0,k) - ik(k^2+4)a^3 T^{(1)}(t_0,k) = f^{(1)}(t_0,k), \quad T^{(1)}(0,k) = 0. \tag{54}$$

$$\dot{T}_1^{(1)}(t_0) = f_1^{(1)}(t_0), \quad T_1^{(1)}(0) = 0, \tag{55}$$

$$\dot{T}_2^{(1)}(t_0) - 8a^3 T_1^{(1)}(t_0) = f_2^{(1)}(t_0), \quad T_2^{(1)}(0) = 0, \tag{56}$$

where

$$f^{(1)}(t_0,k) = \int_{-\infty}^{\infty} R^{(1)}(z,t_0) \psi(z,k) dz, \tag{57}$$

$$f_1^{(1)}(t_0) = -4aa_{t_1} + \int_{-\infty}^{\infty} R^{(1)}(z,t_0) \psi_1(z) dz, \tag{58}$$

$$f_2^{(1)}(t_0)=-4a^3\xi_{t_1}+\int_{-\infty}^{\infty}R^{(1)}(z,t_0)\psi_2(z)dz. \tag{59}$$

The solution of Eq. (54) can also be derived in a standard method:

$$T^{(1)}(t_0,k)=\int_0^{t_0}d\tau\, f^{(1)}(\tau,k)\exp[ik(k^2+4)a^3(t_0-\tau)]. \tag{60}$$

We have seen that, in the case of time-independent perturbations, the secularity condition is $f_1^{(1)}=f_2^{(1)}=0$ (which just determines the time dependence of the two soliton parameters). Although we wonder whether it is always true for time-dependent perturbations, we assume that Eqs. (42) and (43) given in Sec. IV still apply. Of course, for the time-dependent perturbations, they should be replaced by

$$f_1^{(1)}(t_0)=f_2^{(1)}(t_0)=0, \tag{61}$$

which still leads to $T_1^{(1)}(t_0)=T_2^{(1)}(t_0)=0$ and

$$a_{t_1}=\frac{1}{4a}\int_{-\infty}^{\infty}R^{(1)}(z,t_0)\psi_1(z)dz, \tag{62}$$

$$\xi_{t_1}=\frac{1}{4a^3}\int_{-\infty}^{\infty}R^{(1)}(z,t_0)\psi_2(z)dz. \tag{63}$$

From Eqs. (33) and (60), we then obtain the first-order correction:

$$u^{(1)}(z,t_0)=\mathrm{P}\int_0^{t_0}d\tau\int_{-\infty}^{\infty}dk\, f^{(1)}(\tau,k)\times\exp[ik(k^2+4)a^3(t-\tau)]\phi(z,k). \tag{64}$$

Substituting Eq. (57) into Eq. (64), we obtain the first-order Green's function for time-dependent perturbations:

$$u^{(1)}(z,t_0)=\int_0^{t_0}d\tau\int_{-\infty}^{\infty}dz'R^{(1)}(z',t_0)G^{(1)}(z,z';t_0,\tau), \tag{65}$$

$$G^{(1)}(z,z';t_0,\tau)=\mathrm{P}\int_{-\infty}^{\infty}dk\,\exp[ik(k^2+4)\times a^3(t_0-\tau)]\phi(z,k)\psi(z',k). \tag{66}$$

It must be pointed out that although Eq. (61) [which corresponds to the secularity conditions (42) and (43)] is suggested as an assumption, it is easy to check that the solution u obtained under this assumption really satisfies the perturbed KdV equation and the appropriate initial condition

$$u_t-4a^3u_z+6auu_z+a^3u_{zzz}=\epsilon R(z,t),$$
$$u(z,0)=2a^2\,\mathrm{sech}^2\mathcal{Z}. \tag{67}$$

In fact, up to first-order approximation, we have

$$u_t=u_{t_0}+\epsilon(u_{t_1}^{(0)}+u_{t_0}^{(1)}). \tag{68}$$

According to Eqs. (13) and (62)–(64), it follows that

$$u_{t_1}^{(0)}=4aa_{t_1}\phi_1(z)+4a^3\xi_{t_1}\phi_2(z)=\int_{-\infty}^{\infty}R^{(1)}(z',t_0)[\phi_1(z)\psi_1(z')+\phi_2(z)\psi_2(z')]dz', \tag{69}$$

$$u_{t_0}^{(1)}=\mathrm{P}\int_0^{t_0}d\tau\int_{-\infty}^{\infty}dk\,ik(k^2+4)a^3f^{(1)}(\tau,k)\times\exp[ik(k^2+4)a^3(t-\tau)]\phi(z,k)+\mathrm{P}\int_{-\infty}^{\infty}dk\,f^{(1)}(t_0,k)\phi(z,k), \tag{70}$$

$$\begin{aligned}-4a^3u_z^{(1)}+6au^{(0)}u_z^{(1)}+6au_z^{(0)}u^{(1)}+a^3u_{zzz}^{(1)}&=\mathrm{P}\int_0^{t_0}d\tau\int_{-\infty}^{\infty}dk\,f^{(1)}(\tau,k)\times\exp[ik(k^2+4)a^3(t_0-\tau)]a^3\hat{L}\phi(z,k)\\&=-\mathrm{P}\int_0^{t_0}d\tau\int_{-\infty}^{\infty}dk[ik(k^2+4)a^3]f^{(1)}(\tau,k)\times\exp[ik(k^2+4)a^3(t_0-\tau)]\phi(z,k).\end{aligned} \tag{71}$$

Consequently, the left-hand side of the first equation in Eq. (67) becomes

$$u_t-4au_z+6auu_z+a^3u_{zzz}=\epsilon\int_{-\infty}^{\infty}R^{(1)}(z',t_0)\sum_{j=1}^{2}\phi_j(z)\psi_j(z')dz'+\epsilon\int_{-\infty}^{\infty}dk\,f^{(1)}(t_0,k)\phi(z,k). \tag{72}$$

Noting that up to first-order approximation $R(z,t)=R^{(1)}(z,t_0)$, and employing Eq. (57), we finally obtain

$$\begin{aligned}u_t-4a^3u_z+6auu_z+a^3u_{zzz}&=\epsilon\int_{-\infty}^{\infty}\left[\mathrm{P}\int_{-\infty}^{\infty}dk\,\phi(z,k)\psi(z',k)+\sum_{j=1}^{2}\phi_j(z)\psi_j(z')\right]dz'\\&=\epsilon\int_{-\infty}^{\infty}R^{(1)}(z',t_0)\,\delta(z-z')dz'\\&=\epsilon R^{(1)}(z,t_0).\end{aligned} \tag{73}$$

It is clear from Eq. (65) that $u^{(1)}(z,0)=0$, and thus $u(z,t)$ satisfies the initial condition $u(z,0)=2a^2\,\mathrm{sech}^2z$.

VI. DAMPING KdV EQUATION

As an important example, we consider the damping KdV equation in which $R[u]=-u$,

$$u_t+6uu_x+u_{xxx}=-\epsilon u. \tag{74}$$

At first, the time dependence of the soliton parameters can be easily obtained from Eqs. (44) and (45):

$$a_{t_1}=-\frac{1}{4a}\int_{-\infty}^{\infty}2a^2\,\mathrm{sech}^2 z\,\mathrm{sech}^2 z\,dz=-2a/3, \tag{75}$$

$$\xi_{t_1}=-\frac{1}{4a^3}\int_{-\infty}^{\infty}2a^2\,\mathrm{sech}^2 z(\tanh z+z\,\mathrm{sech}^2 z)dz=0. \tag{76}$$

It follows from Eq. (75) immediately that

$$a=a_0\exp(-2t_1/3), \tag{77}$$

which means that the height of the soliton dampens with time exponentially. Next, we calculate the first-order correction given in Eq. (47), where $f^{(1)}(k)$ can be calculated from Eq. (35),

$$f^{(1)}(k)=\sqrt{2\pi}a^2k/3\,\sinh\frac{\pi}{2}k. \tag{78}$$

Substituting Eq. (78) into Eq. (47), we obtain immediately

$$u^{(1)}(z,t_0)=\frac{\sqrt{2\pi}}{3ia}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{1-\exp[ik(k^2+4)a^3t_0]}{(k^2+4)\sinh(\pi k/2)}\,\phi(z,k)dk. \tag{79}$$

We have seen from Eqs. (75) and (76) that the time dependence of the soliton parameters is just the same as that obtained by the other methods [1]. We would like to point out that this is true for the correction $u^{(1)}$ also. At first let us compare it with that derived by Herman [13,14]. To see this, we note that in Eq. (79), parameter a should be taken as a constant up to first-order approximation, and we make the following transformation; $a=\eta_0$, $z=\varphi$, $k=2\lambda/\eta_0$, and replace t_0 by t. Then $\phi(z,k)$ is related to function $\Phi^A(\varphi,t;\lambda)$ used in Refs. [13, 14] by

$$\phi(z,k)=\frac{-i}{2\sqrt{2\pi}\lambda}\,\frac{\lambda+i\eta_0}{\lambda-i\eta_0}\exp(-8i\lambda\eta_0^2t)\Phi^A(\varphi,t;\lambda). \tag{80}$$

Substituting Eq. (80) into Eq. (79), and through some calculations, we get

$$u^{(1)}=\frac{\sqrt{2\pi}}{3ia}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{1-\exp[ik(k^2+4)a^3t]}{(k^2+4)\sinh(\pi k/2)}\,\phi(z,k)dk$$
$$=\mathrm{P}\int_{-\infty}^{\infty}\frac{\exp(-8i\lambda\eta_0^2t)-\exp(8i\lambda^3t)}{12\lambda(i\lambda+\eta_0)^2\sinh(\pi\lambda/\eta_0)}\,\Phi^A(\varphi,t;\lambda)d\lambda. \tag{81}$$

It is somewhat different from u_1 given in Eq. (73) in Ref. [13] by a factor $-(\lambda^2+\eta_0^2)$ in the integrand. This difference is caused by some errors in the calculation in Ref. [13]. In fact, if we start from Eq. (B1) in Ref. [13], we obtain also our u_1.

Moreover, our $u^{(1)}$ given by Eq. (79) is consistent with that obtained by the inverse scattering perturbation theory [1]. To show this, we neglect the slow time dependence of the soliton parameters in the first-order correction, namely, we let $t_0=t$, $a=a_0$, and $\xi=\xi_0+4a^2t$ in Eq. (79), up to the first-order approximation. Then we take the same approximation as Ref. [1]: (a) Neglect the slow z dependence in $\tanh z$, i.e., we let $\tanh^2 z\approx 1$ in the integrand at the right-hand side of Eq. (79); (b) Since the factor e^{ikz} in the integrand of Eq. (79) oscillates rapidly for large k, we expect that the dominant contribution to the integral will come from the region near $k=0$. Thus the terms with higher powers of k can be neglected. After doing these, $u^{(1)}$ is simplified into

$$u^{(1)}(x,t)=\frac{1}{3a_0}\,\mathrm{P}\int_{-\infty}^{\infty}\frac{dk}{2\pi i}\,\frac{e^{2ik(x-\xi_0)}}{k}\,(e^{8ik^2t}-e^{-8ika_0^2t}). \tag{82}$$

Consequently,

$$u_x^{(1)}=\frac{1}{6\pi a_0}\int_{-\infty}^{\infty}d\kappa\,e^{i\kappa(s-\xi_0)}(e^{i\kappa^3t}-e^{-4ia_0^2\kappa t})$$
$$=\frac{1}{3a_0(3t)^{1/3}}\,\mathrm{Ai}\left[\frac{x-\xi_0}{(3t)^{1/3}}\right]-\delta(x-\xi_0-4a_0^2t), \tag{83}$$

$$u^{(1)}(x,t)=\frac{1}{3a_0}\int_{-\infty}^{(x-\xi_0)/(3t)^{1/3}}d\kappa\,\mathrm{Ai}(\kappa)-\theta(x-\xi_0-4a_0^2t), \tag{84}$$

where $\theta(x)$ and $\mathrm{Ai}(x)$ are the step function and Airy function, respectively. Equations (82)–(84) are all the same as those obtained by the inverse scattering perturbation theory given by Eqs. (9.2.31)–(9.2.33) in Ref. [1].

VII. KdV-BURGERS EQUATION

As another example, we consider KdV-Burgers equation

$$u_t+6uu_x+u_{xxx}=\epsilon u_{xx}. \tag{85}$$

Here $R[u]=u_{zz}$ and thus

$$R^{(1)}[u^{(0)}]=u_{xx}^{(0)}=8a^4\,\mathrm{sech}^2 z-12a^4\,\mathrm{sech}^4 z. \tag{86}$$

Inserting Eq. (86) into Eqs. (44) and (45), we get

$$a_t=\frac{\epsilon}{4a}\int_{-\infty}^{\infty}(8a^4\,\mathrm{sech}^2 z-12a^4\,\mathrm{sech}^4 z)\mathrm{sech}^2 z\,dz$$
$$=-8\epsilon a^3/15, \tag{87}$$

$$\xi_t=4a^2+\frac{\epsilon}{4a^3}\int_{-\infty}^{\infty}(8a^4\,\mathrm{sech}^2 z-12a^4\,\mathrm{sech}^4 z)(\tanh z$$
$$+z\,\mathrm{sech}^2 z)dz=4a^2, \tag{88}$$

respectively. It follows that

$$a=a_0/(1+16\epsilon a_0^2 t/15)^{1/2}, \tag{89}$$

$$\xi=\xi_0+15\ \ln(1+16\epsilon a_0^2 t/15)/4\epsilon, \tag{90}$$

which are the same as those derived by other methods [11]. To find out the first-order correction, let us calculate the expansion coefficient corresponding to the continuous spectrum:

$$\begin{aligned} f^{(1)}(k)&=\int_{-\infty}^{\infty}(8a^4\ \mathrm{sech}^2 z-12a^4\ \mathrm{sech}^4 z)\psi(z,k)dz\\ &=24\pi\sqrt{2\pi}a^4(2k/15-k^5/100)/(k^2+4)\sinh(\pi k/2)\\ &\quad+4\sqrt{2\pi}a^4 k/3\ \sinh(\pi k/2). \end{aligned} \tag{91}$$

The correction is obtained by inserting Eq. (86) into Eq. (35). It is interesting that under the same approximation as that introduced for the damping KdV equation, we have

$$\begin{aligned} u^{(1)}(x,t)=&-\frac{32}{15}a_0\ \mathrm{P}\int_{-\infty}^{\infty}\frac{dk}{2\pi i}\frac{e^{2ik(x-\xi_0)}}{k}\\ &\times(e^{8ik^3 t}-e^{-8ika_0^2 t}), \end{aligned} \tag{92}$$

which is proportional to the correction for the damping KdV equation [see Eq. (82)]. We should note that they have an opposite sign.

APPENDIX A: DERIVATION OF EIGENFUNCTIONS $\phi(z,k)$ AND $\psi(z,k)$

Let us consider the eigenvalue problem

$$\hat{L}\phi=\lambda\phi, \tag{A1}$$

where $\hat{L}=d^3/dz^3+(12\ \mathrm{sech}^2 z-4)d/dz-24\ \tanh z\ \mathrm{sech}^2 z$. Obviously, $\hat{L}\to d^3/dz^3-4d/dz$ as $z\to\pm\infty$. We assume that

$$\phi(z,k)=\theta(z,k)e^{ikz}, \tag{A2}$$

where $\theta(z,k)$ is supposed to have the asymptotic behavior: $\theta(z,k)\to$const as $z\to\pm\infty$. Then the asymptotic equation of Eq. (A1) leads to $\lambda=-ik(k^2+4)$. Inserting Eq. (A2) into Eq. (A1), one obtains an ordinary differential equation for θ,

$$\hat{L}\theta+ik(3d^2/dz^2+12\ \mathrm{sech}^2 z)\theta-3k^2 d\theta/dz=0. \tag{A3}$$

To determine $\theta(z,k)$, we expand it into a power series of ik whose coefficients $\theta^{(n)}(z)$ are functions of z only,

$$\theta=\theta^{(0)}(z)+ik\theta^{(1)}(z)+(ik)^2\theta^{(2)}(z)+\cdots . \tag{A4}$$

Substituting Eq. (A4) into Eq. (A3), and equating the coefficients of each power of ik, one obtains the following recursion equations:

$$\hat{L}\theta^{(0)}=0, \tag{A5}$$

$$\hat{L}\theta^{(1)}=-3\theta_{zz}^{(0)}-12\ \mathrm{sech}^2 z\,\theta^{(0)}, \tag{A6}$$

$$\hat{L}\theta^{(2)}=-3\theta_{zz}^{(1)}-12\ \mathrm{sech}^2 z\,\theta^{(1)}-3\theta_z^{(0)}, \tag{A7}$$

$$\hat{L}\theta^{(3)}=-3\theta_{zz}^{(2)}-12\ \mathrm{sech}^2 z\,\theta^{(2)}-3\theta_z^{(1)}\dots . \tag{A8}$$

It is easy to solve Eqs. (A5)–(A8) successively, and then we get the following special solution:

$$\theta^{(0)}=c\ \tanh z\ \mathrm{sech}^2 z,\quad \theta^{(1)}=c(\tfrac{1}{2}-\mathrm{sech}^2 z),$$

$$\theta^{(2)}=-\frac{c}{2}\tanh z,$$

$$\theta^{(3)}=\tfrac{1}{8}c,\quad \theta^{(n)}=0\quad \text{for } n\geqslant 4, \tag{A9}$$

where the arbitrary constant c is determined by the normalization condition as $c=8i/k(k^2+4)\sqrt{2\pi}$. Consequently we obtain

$$\begin{aligned} \phi(z,k)&=\frac{1}{k(k^2+4)\sqrt{2\pi}}[k(k^2-4)+4ik^2\ \tanh z\\ &\quad+8k\ \mathrm{sech}^2 z+8i\ \tanh z\ \mathrm{sech}^2 z]e^{ikz}\\ &=\frac{1}{k(k^2+4)\sqrt{2\pi}}[k(k^2+4)+4i(k^2+2)\tanh z\\ &\quad-8k\ \tanh^2 z-8i\ \tanh^3 z]e^{ikz}. \end{aligned} \tag{A10}$$

The eigenfunction $\psi(z,k)$ of $\hat{L}^\dagger$ can be derived in the same way.

APPENDIX B: COMPLETENESS OF SETS $\{\phi\}$ AND $\{\psi\}$

We start from the integral

$$\mathrm{P}\int_{-\infty}^{\infty}\phi(z,k)\psi(z',k)dk=\mathrm{P}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}e^{ik(z-z')}+f(k)\right]dk, \tag{B1}$$

where $f(k)$ is defined by

$$f(k)=\phi(z,k)\psi(z',k)-\frac{1}{2\pi}e^{ik(z-z')}. \tag{B2}$$

Inserting Eqs. (22) and (24) into Eq. (B2), $f(k)$ is rewritten as

$$\begin{aligned} f(k)&=\frac{1}{2\pi}e^{ik(z-z')}g(k)\\ &=\frac{1}{2\pi}e^{ik(z-z')}[g_1(k)+g_2(k)+g_1(k)g_2(k)], \end{aligned} \tag{B3}$$

with

$$g_1(k)=\frac{4}{k^2+4}(ik\ \tanh z-2\ \tanh^2 z+2i\ \tanh z\ \mathrm{sech}^2 z/k), \tag{B4}$$

$$g_2(k)=-\frac{4}{k^2+4}(ik\ \tanh z'+\tanh^2 z'+1). \tag{B5}$$

It is easy to see that $f(k)$ as a function of complex variable k is analytical everywhere except for a simple pole $k_0=0$ and two double poles $\pm 2i$, at which the residues of $f(k)$ can be derived in a standard way:

$$\text{Res } f(0)=\lim_{k\to 0} k f(k)=-i \text{ tanhz sech}^2 z(1-\text{sech}^2 z')/\pi, \tag{B6}$$

$$\begin{aligned}\text{Res } f(\pm 2i)&=\lim_{k\to\pm 2i}\frac{d}{dk}[(k\mp 2i)f(k)]\\&=\frac{i}{2\pi}\text{ tanhz sechz(sechz}' e^{\pm z'}\pm z' \text{ sech}^2 z'\\&\quad\mp z \text{ sech}^2 z')\pm\frac{i}{2\pi}e^{\mp z}\text{ sech}^3 z \text{ sech}^2 z'.\end{aligned} \tag{B7}$$

Now, let us return to Eq. (B1). Obviously the first term of the integrand at the right-hand side of Eq. (B1) contributes $\delta(z-z')$. The integral of the second term can be calculated with the aid of the residue theorem. Since $|g(k)|\to 0$ as $|k|\to\infty$, according to Jordan's lemma, it follows that

$$\text{P}\int_{-\infty}^{\infty} f(k)dk=\pm 2\pi i[\tfrac{1}{2}\text{ Res } f(0)+\text{Res } f(\pm 2i)], \tag{B8}$$

where, the upper or lower symbol corresponds to $(z-z')>0$ or $(z-z')<0$, respectively. Substituting Eqs. (B6) and (B7) into Eq. (B8), we get immediately

$$\begin{aligned}\int_{-\infty}^{\infty} f(k)dk&=-(\text{sech}^2 z-z \text{ tanhz sech}^3)\text{sech}^2 z'\\&\quad-\text{tanhz sech}^2 z(\text{tanhz}'+z' \text{ sech}^2 z')\\&=-[\phi_1(z)\psi_1(z')+\phi_2(z)\psi_2(z')].\end{aligned} \tag{B9}$$

Finally we obtain the completeness relation from Eqs. (B1) and (B9),

$$\int_{-\infty}^{\infty}\phi(z,k)\psi(z',k)dk+\sum_{j=1}^{2}\phi_j(z)\psi_j(z')=\delta(z-z'). \tag{B10}$$

APPENDIX C: ORTHOGONALITY OF SETS $\{\phi\}$ AND $\{\psi\}$

Before the proof of the orthogonality relations, we must derive some useful integral formulas. At first let us calculate the following integral by the aid of the residue theorem:

$$I_1=\int_{-\infty}^{\infty} e^{i(k-k')z}\text{ tanhz } dz. \tag{C1}$$

To do this, we consider a complex integral along a closed path c in plane $\zeta=z+i\eta$ as follows:

$$\oint_c f_1(\zeta)d\zeta=\oint_c e^{i(k-k')\zeta}\text{ tanh}\zeta\, d\zeta. \tag{C2}$$

Since the factor $\tanh\xi$ in the integrand is a periodic function with an imaginary period $i\pi$, we choose c to be the boundary of a rectangular region infinitely long: $-\infty<z<\infty$, $0\leqslant\eta\leqslant\pi$. In this region the integrand is analytic except for a simple pole $\zeta_0=i\pi/2$. We note that $\epsilon^{i(k-k)\zeta}$ oscillates rapidly for large z. Then the integral along the two straight line segments: $z=\pm\infty$, $0\leqslant\eta\leqslant\pi$ should be zero. It follows that

$$\oint_c f_1(\zeta)d\zeta=[1-e^{-(k-k')\pi}]I_1. \tag{C3}$$

On the other hand, according to the residue theorem,

$$\oint_c f_1(\zeta)d\zeta=2\pi i \text{ Res } f_1(\zeta_0). \tag{C4}$$

The residue is easily obtained by the standard method

$$\text{Res } f_1(\zeta_0)=\lim_{\zeta\to\zeta_0}(\zeta-\zeta_0)f_1(\zeta)=e^{-(k-k')\pi/2}. \tag{C5}$$

Comparing Eqs. (C3), (C4), and (C5), we get immediately

$$I_1=\int_{-\infty}^{\infty} e^{i(k-k')z}\tanh z\, dz=i\pi/\sinh\tfrac{1}{2}\pi(k-k'). \tag{C6}$$

Starting from Eq. (C6), and employing the techniques of integration by parts repeatedly, we can perform the following integrals successively:

$$I_2=\int_{-\infty}^{\infty} e^{i(k-k')z}\frac{1}{\cosh^2 z}dz=\pi(k-k')/\sinh\tfrac{1}{2}\pi(k-k'), \tag{C7}$$

$$\begin{aligned}I_3&=\int_{-\infty}^{\infty} e^{i(k-k')z}\frac{\text{sinhz}}{\cosh^3 z}dz\\&=\frac{i}{2}\pi(k-k')^2/\sinh\tfrac{1}{2}\pi(k-k'),\end{aligned} \tag{C8}$$

$$\begin{aligned}I_4&=\int_{-\infty}^{\infty} e^{i(k-k')z}\frac{1}{\cosh^4 z}dz=\tfrac{1}{6}\pi[(k-k')^3\\&\quad+4(k-k')]/\sinh\tfrac{1}{2}\pi(k-k'),\end{aligned} \tag{C9}$$

$$\begin{aligned}I_5&=\int_{-\infty}^{\infty} e^{i(k-k')z}\frac{\text{sinhz}}{\cosh^5 z}dz=\frac{i}{24}[(k-k')^4\\&\quad+4(k-k')^2]/\sinh\tfrac{1}{2}\pi(k-k').\end{aligned} \tag{C10}$$

Now we return to the orthogonality relations. As an example, let us perform the following integral in detail:

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(z,k)\psi(z,k')dz&=\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}e^{i(k-k')z}+f(z)\right]dz\\&=\delta(k-k')+\int_{-\infty}^{\infty} f(z)dz,\end{aligned} \tag{C11}$$

where the function $f(z)$ is defined as

$$f(z)=\phi(z,k)\psi(z,k')-\frac{1}{2\pi}e^{i(k-k')z}. \quad \text{(C12)}$$

Equations (22) and (24) can be rewritten into

$$\phi(z,k)=ce^{ikz}\left[k(k^2-4)+4ik^2\ \tanh z+\frac{8k}{\cosh^2 z}+8i\frac{\sinh z}{\cosh^3 z}\right], \quad \text{(C13)}$$

$$\psi(z,k')=c'e^{-ik'z}\left[(k'^2-4)-4ik'\ \tanh z+\frac{4}{\cosh^2 z}\right], \quad \text{(C14)}$$

respectively. In Eqs. (C13) and (C14), $c=1/\sqrt{2\pi}k(k^2+4)$ and $c'=1/\sqrt{2\pi}k(k^2+4)$ are normalization constants. Substituting Eqs. (C13) and (C14) into Eq. (C11), we get

$$\int_{-\infty}^{\infty}f(z)dz=cc'[-4i(k-k')(kk'+4)I_1+4(k^3+4k^2k'+2kk'^2-12k+8k')I_2+4i(4k^2-8kk'+2k'^2-8)I_3+32(k-k')I_4+32iI_5]. \quad \text{(C15)}$$

Inserting Eqs. (C6)–(C10) into Eq. (C15), we get $\int_{-\infty}^{\infty}f(z)dz=0$ through a series of calculations. Finally we obtain from Eq. (C11) that

$$\int_{-\infty}^{\infty}\phi(z,k)\psi(z,k')dz=\delta(k-k'). \quad \text{(C16)}$$

Thus Eq. (27) has been proved. The orthogonality relations (28) can be derived in the same way, while Eq. (29) is easily obtained by a straightforward calculation.

[1] G. L. Lamb. *Elements of Soliton Theory* (Wiley, New York, 1980) pp. 259–278.
[2] D. J. Kaup, SIAM J. Appl. Math. **31**, 121 (1976).
[3] D. J. Kaup and A. C. Newell, Proc. R. Soc. London Ser. A **361**, 413 (1978).
[4] V. I. Karpman, Pis'ma Zh. Eksp. Teor. Fiz **25**, 296 (1977) [JEPT Lett. **25**, 271 (1977)].
[5] V. I. Karpman and E. M. Maslov, Zh. Eksp. Teor. Fiz. **73**, 537 (1977) [Sov. Phys. JETP **46**, 281 (1977)].
[6] K. A. Gorshkov, L. A. Ostrovskii, and E. N. Pelinovsky, Proc. IEEE **82**, 1511 (1974).
[7] K. A. Gorshkov and L. A. Ostrovskii, Physica D **3**, 428 (1981).
[8] M. B. Fogel *et al.*, Phys. Rev. B **15**, 1578 (1977).
[9] G. Reinisch and C. Fernandez, Phys. Rev. B **24**, 835 (1981).
[10] V. I. Karpman, Phys. Lett. A **71**, 163 (1979).
[11] M. Tanaka, J. Phys. Soc. Jpn. **49**, 807 (1980).
[12] S. Watanabe, J. Phys. Soc. Jpn. **45**, 276 (1978).
[13] R. L. Herman, J. Phys. A **23**, 2327 (1990).
[14] R. L. Herman, J. Phys. A **23**, 1063 (1990).
[15] J. P. Keener and D. W. McLaughlin, Phys. Rev. A **16**, 777 (1977).
[16] D. J. Kaup, Phys. Rev. B **27**, 6787 (1983).
[17] D. J. Kaup, Phys. Rev. B **29**, 1072 (1984).
[18] A. R. Bishop *et al.*, Physica D **1**, 1 (1980).
[19] K. Ko and H. H. Keuhl, Phys. Rev. Lett. **40**, 233 (1978).
[20] Y. Kodama and M. J. Ablowitz, Stud. Appl. Math. **64**, 225 (1981).